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INFLUENCE OF THE ACCOMMODATION COEFFICIENT ON THE HEAT TRANSFER IN A RAREFIED GAS

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Abstract—Linearized heat transfer between parallel plates and concentric cylinders is considered for inverse Knudsen numbers ranging from 0 to 10, and arbitrary accommodation coefficients. The Bhatnagar, Gross and Krook model is used and transformed into a system of integral equations.

These equations are solved by a variational method, previously applied to simpler situations. Comparisons are made with experimental results and previously available calculations.

NOMENCLATURE

C_1, C_2, surface of the inner (outer) cylinder;

d, distance between the plates [cm];

h, perturbation of the basic Maxwellian;

$\mathbf{n}$, inward normal unit vector;

p, pressure [g/cm s^2];

$P_{ij}^{(in)}$, $p_{ij}^{(out)}$, contribution to stress tensor from arriving (leaving) molecules [g/cm s^2];

q, relevant component of the heat flux vector [g/s^3];

q_{fm}, free-molecular value of q[g/s^3];

$q^{(in)}$, $q^{(out)}$, contribution to heat flux vector from arriving (leaving) molecules [g/s^3];

R, gas constant [cm^2/degK s^2];

r, nondimensional radial coordinate in cylindrical system;

r_1, r_2, nondimensional radius of the inner (outer) cylinder;

T, unperturbed temperature (plane case) [degK];

T_1, T_2, temperature of the upper (lower) plate (plane case), of the inner (outer) cylinder (cylindrical case) [degK];

T_n, transcendental functions defined by equation (2.15);

$\mathbf{u}^{(in)}$, $\mathbf{u}^{(out)}$, contribution to mass velocity from arriving (leaving) molecules [cm/s];

$\mathbf{x}$, space vector [cm];

x, nondimensional coordinate in plane case.

Greek symbols

α, $= \alpha_E$;

α_M, α_E, accommodation coefficient for momentum (energy);

δ, $= d/[\theta(2RT)^{\frac{1}{2}}]$;

ε_i, perturbations of density and temperature ($i = 1, 2$);

θ, mean free time [s];

μ, viscosity coefficient [g/cm s];

ξ, modulus of $\boldsymbol{\xi}$;

$\boldsymbol{\xi}$, ratio of the molecular velocity vector to $(2RT)^{\frac{1}{2}}$;

ξ_x, x component of $\boldsymbol{\xi}$;

ρ, average density (plane case) [g/cm^3];

ρ_1, ρ_2, density of the re-emitted molecules at the inner (outer) cylinder [g/cm^3];

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τ, $= (T_1 - T_2)/2T$, (plane case), $= (T_1 - T_2)/T_2$, (cylindrical case);

α_{ik}, γ^*, $\mathbf{v}^*$, τ^*, a_i, c_i, auxiliary quantities.

1. INTRODUCTION

IN A PREVIOUS paper [1] we treated the problem of heat transfer in a rarefied gas between parallel plates under the assumption of a complete accommodation of the gas to the state of the boundary. The results, although interesting, can hardly be compared with experimental data, since the accommodation coefficient for usual gas–surface interactions is sensibly smaller than 1 (it ranges from about 0·3 to about 0·8). Also, a typical experimental arrangement for measurement of heat transfer shows a cylindrical rather than a plane geometry, although the latter has been recently used by Teagan [2]. In order to bring the theory into a form comparable with experimental result, we wish to consider the problem of heat transfer between both parallel plates and concentric cylinders with arbitrary accommodation coefficients. As in the previous paper, our aim is to give a very accurate answer for the heat transfer according to the Bhatnagar, Gross and Krook (BGK) model in order to assess the capability of the latter for describing the transition regime in agreement with experimental data. The method to be used in our calculation is the variational procedure proposed in [3] and used in [3, 1, 4]. The previous results [1] show that the values for heat transfer as given by this method are practically indistinguishable from those obtained by an accurate numerical solution of the BGK equation (the disagreement is less than 1 per cent). Because of this fact and of the much smaller amount of computing time required by the variational procedure, we have not computed an accurate numerical solution of the BGK model but assume, on the basis of the previous experience [1, 3, 4] that the present results differ less than 1 per cent from the exact BGK values.

2. BASIC EQUATIONS

(a) *Boundary conditions*

The boundary conditions to be matched with the Boltzmann equation have been recently discussed [5]. In particular, it has been shown that one can construct a hierarchy of models capable of approximating any experimental situation with arbitrary accuracy. One of the simplest models furnishes for the linearized Boltzmann equation

$$h(\mathbf{x}, \boldsymbol{\xi}) = \gamma^* + \tau^* \xi^2 + \mathbf{v}^* \cdot \boldsymbol{\xi} \qquad (\mathbf{x} \in \partial R; \boldsymbol{\xi}\mathbf{n} > 0) \tag{2.1}$$

where h is the perturbation of the basic Maxwellian characterized by the average density ρ and temperature T, $\boldsymbol{\xi}$ is the molecular velocity vector measured in $(2RT)^{\frac{1}{2}}$ units, $\mathbf{x}$ the position vector, ∂R the boundary of the region R filled with gas, $\mathbf{n}$ the normal pointing into R, γ^*, τ^*, $\mathbf{v}^*$ certain functionals of the distribution function of the molecules arriving at the wall and of both the accommodation properties and the physical state of the wall. These functionals are determined by the conditions:

$$|\mathbf{n} \cdot \mathbf{u}^{(\text{in})}| = |\mathbf{n} \cdot \mathbf{u}^{(\text{out})}| \tag{2.2}$$

$$(1 - \alpha_M)|n_i p_{iJ}^{(\text{in})}| = |n_i p_{iJ}^{(\text{out})}| - \alpha_M |n_i p_{iJ}^{(w)}| \tag{2.3}$$

$$(1 - \alpha_E)|\mathbf{n} \cdot \mathbf{q}^{(\text{in})}| = |\mathbf{n} \cdot \mathbf{q}^{(\text{out})}| - \alpha_E |\mathbf{n} \cdot \mathbf{q}^{(w)}| \tag{2.4}$$

where $\mathbf{u}^{(\text{in})}$, $\mathbf{u}^{(\text{out})}$, $p_{iJ}^{(\text{in})}$, $p_{iJ}^{(\text{out})}$, $\mathbf{q}^{(\text{in})}$, $\mathbf{q}^{(\text{out})}$, are the contributions to mass velocity, stress tensor and heat flux from arriving and respectively, leaving molecules, α_M and α_E the accommodation coefficients for momentum and energy, while $p_{iJ}^{(w)}$ and $\mathbf{q}^{(w)}$ are the values of $p_{iJ}^{(\text{out})}$ and $\mathbf{q}^{(\text{out})}$ for a completely accommodating wall ($\alpha_E = \alpha_M = 1$). The above equations can be obtained either as particularly simple cases of the above mentioned hierarchy or by linearization of the boundary condition of re-emission according Maxwellian with mass velocity and temperature different from those of the wall, proposed by Welander [6] and Nocilla [7]. Equation (2.2) is simply the condition for the number of molecules to be conserved at the wall, while

equations (2.3) and (2.4) immediately follow from the definitions of α_M and α_E.

In this regard we notice that sometimes [8, 9] momentum and energy (i.e. mass velocity and temperature) are used in place of their fluxes (i.e. stress tensor and heat flux) to define α_M and α_E; in many problems these definitions are equivalent but the one based on fluxes is to be regarded as the correct one in general (one has to compute numbers of molecules arriving at the wall per unit time).

(b) *Plane case*

In the plane case axes are taken with the origin halfway between the parallel plates, which become $x = \pm d/2$. The temperatures of the plates are $T_1 = T + \Delta T$ (at $x = -d/2$) and $T_2 = T - \Delta T$ (at $x = d/2$). The boundary conditions at each wall are those described in Section 1. Because of symmetry reasons $v^* = 0$ and γ^*, τ^* at $x = d/2$ are opposite to the same quantities at $x = -d/2$. Therefore we can write

$$h\left(-\frac{d}{2}\operatorname{sgn}\xi_x, \boldsymbol{\xi}\right) = (\gamma^* + \tau^*\xi^2)\operatorname{sgn}\xi_x \tag{2.5}$$

where γ^* and τ^* are constants to be determined according to equations (2.2) and (2.4). Using the linearized version of the BGK model, the transport equation becomes

$$\theta\xi_x\frac{\partial h}{\partial x} + h = \varepsilon_1(x) + (\xi^2 - \tfrac{3}{2})\,\varepsilon_2(x) \tag{2.6}$$

where θ is the mean free time related to viscosity and pressure by

$$\theta = \frac{3}{2}\frac{\mu}{p} \tag{2.7}$$

and

$$\varepsilon_1(x) = \pi^{-\frac{3}{2}}\iiint \exp\left[-\xi^2\right] h(x, \boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \tag{2.8}$$

$$\varepsilon_2(x) = \tfrac{2}{3}\pi^{-\frac{3}{2}}\iiint \exp\left[-\xi^2\right](\xi^2 - \tfrac{3}{2})\,h(x, \boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}. \tag{2.9}$$

Equation (2.6) can be integrated to give a couple of integral equations for $\varepsilon_1(x)$, $\varepsilon_2(x)$:

$$\boldsymbol{\varepsilon} = \mathfrak{S}\boldsymbol{\varepsilon} + \mathbf{S} \tag{2.10}$$

where $\boldsymbol{\varepsilon}$ and $\mathbf{S}$ are two-dimensional vectors given by

$$\boldsymbol{\varepsilon} = \begin{vmatrix} \varepsilon_1 \\ (3/2)^{\frac{1}{2}}\,\varepsilon_2 \end{vmatrix} \tag{2.11}$$

$$\mathbf{S} = -\pi^{-\frac{1}{2}}\begin{vmatrix} \tau^*\{T_2 + T_0\}^- + \gamma^*\{T_0\}^- \\ (\tfrac{2}{3})^{\frac{1}{2}}\left[\tau^*\{T_4 + \tfrac{1}{2}T_2 + \tfrac{1}{2}T_0\}^- + \gamma^*\{T_2 - \tfrac{1}{2}T_0\}^-\right] \end{vmatrix}. \tag{2.12}$$

Here we have put for any function $f(x)$:

$$\{f\}^- = f(\delta/2 - x) - f(\delta/2 + x) \quad (\delta = d/\theta) \tag{2.13}$$

and measured x in θ units. $\mathfrak{S}$ is a two-by-two matrix operator given by:

$$\mathfrak{S} = \pi^{-\frac{1}{2}}\int_{-\delta/2}^{+\delta/2}\begin{vmatrix} T_{-1} & (\frac{2}{3})^{\frac{1}{2}}(T_1 - \frac{1}{2}T_{-1}) \\ (\frac{2}{3})^{\frac{1}{2}}(T_1 - \frac{1}{2}T_{-1}) & (\frac{2}{3})(T_3 - T_1 + \frac{5}{4}T_{-1}) \end{vmatrix}\cdot \mathrm{d}y \tag{2.14}$$

where the argument of the functions T_n is $|x - y|$. The T_n functions themselves are defined by:

$$T_n(x) = \int_0^\infty t^n \exp\left[-t^2 - \frac{x}{t}\right]\mathrm{d}t. \tag{2.15}$$

Equations (2.2) and (2.4) give two complementary conditions for each wall, i.e. four conditions; if we take into account the symmetry of the problem, we can write just two

conditions, obtained by suitably combining the conditions at the two walls. We obtain

$$\int_{-\delta/2}^{+\delta/2} [\varepsilon_1(x)\{T_0\}^- + \varepsilon_2(x)\{T_2 - \tfrac{1}{2}T_0\}^-]\,dx = -2\tau^*[T_3(\delta) + T_1(\delta) + 1] - 2\gamma^*[T_1(\delta) + \tfrac{1}{2}] \tag{2.16}$$

$$\int_{-\delta/2}^{+\delta/2} [\varepsilon_1(x)\{T_2 + T_0\}^- + \varepsilon_2(x) \times \{T_4 + \tfrac{1}{2}T_2 + \tfrac{1}{2}T_0\}^-]\,dx = -2\gamma^*[T_3(\delta) + T_1(\delta) + 1] - 2\tau^*\left[T_5(\delta) + 2T_3(\delta) + 2T_1(\delta) - \frac{2\alpha - 3}{1 - \alpha}\right] + 2\tau\alpha(1 - \alpha)^{-1} \tag{2.17}$$

where $\tau = \Delta T/T_0$. α is the accommodation coefficient for energy, previously denoted by α_E.

One can easily evaluate the heat flux vector q in terms of α, τ, τ^* and find

$$\frac{q}{q_{fm}} = \frac{1 - \alpha/2}{1 - \alpha}(1 - \tau^*/\tau) \tag{2.18}$$

where q_{fm} is the free-molecular value given by

$$q_{fm} = \rho\alpha\tau\pi^{-\frac{1}{2}}/(2 - \alpha). \tag{2.19}$$

Let us consider now the following functional

$$J(\tilde{\varepsilon}, \tilde{\gamma}^*, \tilde{\tau}^*) = (\tilde{\varepsilon}, \tilde{\varepsilon} - \mathfrak{S}\tilde{\varepsilon} - 2\mathbf{S}) + \alpha_{33}\tilde{\gamma}^{*2} + \alpha_{44}\tilde{\tau}^{*2} + 2\alpha_{34}\tilde{\gamma}^*\tilde{\tau}^* - 2c_4\tilde{\tau}^* \tag{2.20}$$

where

$$\left.\begin{aligned} \alpha_{33} &= \pi^{-\frac{1}{2}}[1 + 2T_1(\delta)] \\ \alpha_{44} &= \pi^{-\frac{1}{2}}\left[12T_1(\delta) + \delta T_2(\delta) + 4\delta T_0(\delta) - 2\frac{2\alpha - 3}{1 - \alpha}\right] \\ \alpha_{34} &= \pi^{-\frac{1}{2}}[2 + \delta T_0(\delta) + 4T_1(\delta)] \\ c_4 &= 2\pi^{-\frac{1}{2}}\frac{\alpha\tau}{1 - \alpha} \end{aligned}\right\} \tag{2.21}$$

and

$$(f, g) = \int_{-\delta/2}^{+\delta/2} f(x)\,g(x)\,dx. \tag{2.22}$$

It is easily verified that J attains its minimum value when $\tilde{\varepsilon} = \varepsilon$, $\tilde{\gamma}^* = \gamma^*$, $\tilde{\tau}^* = \tau^*$ where ε, γ^*, τ^* constitute the solution of the system formed by equations (2.10, 2.16, 2.17).

The corresponding minimum value of J is easily shown to be related to the ratio q/q_{fm} given by equation (2.18) as follows:

$$\frac{q}{q_{fm}} = \frac{1 - \alpha/2}{1 - \alpha}\left[1 + \frac{\min J}{\tau c_4}\right]. \tag{2.23}$$

Therefore even a rough estimate for ε, γ^* and τ^* can give very accurate answer for the heat flux.

(c) *Cylindrical case*

In the cylindrical case let r_1, r_2, $(r_2 > r_1)$ denote the radii of the cylinders (measured in θ units) at the temperatures T_1 and T_2. Let us assume a boundary condition of complete accommodation at the outer cylinder C_2 and denote by α the accommodation coefficient of the inner cylinder C_1. If we linearize the distribution function about the Maxwellian of the molecules leaving the outer cylinder and adopt the BGK model, we have to solve the following equation

$$\boldsymbol{\xi} \cdot \frac{\partial h}{\partial \mathbf{x}} + h = \varepsilon_1(\mathbf{x}) + (\xi^2 - \tfrac{3}{2})\,\varepsilon_2(\mathbf{x}) \tag{2.24}$$

where ε_1 and ε_2 are related to h by equations (2.8) and (2.9) and θ units are used throughout.

The boundary conditions for $h(\mathbf{x}, \boldsymbol{\xi})$ are:

$$h = \begin{cases} 0 & \mathbf{x} \in C_2 \quad \boldsymbol{\xi} \cdot \mathbf{n} < 0 \\ \gamma^* + \tau^* \xi^2 & \mathbf{x} \in C_1 \quad \boldsymbol{\xi} \cdot \mathbf{n} > 0. \end{cases} \tag{2.25}$$

Equation (2.24) can be integrated to give a couple of integral equations for $\varepsilon_1(r)$, $\varepsilon_2(r)$, where r is the radial coordinate in a cylindrical system; these equations can be written in the form (2.10) where ε is given by equation (2.11) while $\mathfrak{S}$ and $\mathbf{S}$ are given by:

$$\mathbf{S} = \frac{1}{\pi}\begin{vmatrix} \int_{-\varphi_0}^{+\varphi_0} \{\gamma^* T_1 + \tau^*[T_3 + \tfrac{1}{2}T_1]\}\,d\varphi & \\ (\tfrac{2}{3})^{\frac{1}{2}} \int_{-\varphi_0}^{+\varphi_0} \{\gamma^*[T_3 - T_1] + \tau^*[T_5 - \tfrac{1}{2}T_3]\}\,d\varphi & \end{vmatrix} \tag{2.26}$$

$$\mathfrak{S} = \frac{1}{\pi}\int_{\Sigma(x)} \begin{vmatrix} T_0 & (\tfrac{2}{3})^{\frac{1}{2}}(T_2 - T_0) \\ (\tfrac{2}{3})^{\frac{1}{2}}(T_2 - T_0) & \tfrac{2}{3}(T_4 - 2T_2 + \tfrac{3}{2}T_0) \end{vmatrix} \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{2.27}$$

where the arguments of the T_n functions are

$$|\mathbf{r} - \mathbf{r}_1| = r\cos\varphi - \sqrt{(r^2\cos^2\varphi - r^2 + r_1^2)} \tag{2.28}$$

in equation (2.26), and

$$|\mathbf{r} - \mathbf{r}'| = r\cos\varphi - \sqrt{(r^2\cos^2\varphi - r^2 + r'^2)} \tag{2.29}$$

in equation (2.27).

Here φ denotes an integration variable of angular nature, $\varphi_0 = \sin^{-1}(r_1/r)$ and the integration with respect to $r'(d\mathbf{r}' = r'\,dr'\,d\varphi)$ is over the region between the cylinders which can be reached from $\mathbf{x}$ by straight lines nonintersecting boundaries.

Equations (2.2) and (2.4) give two complementary conditions:

$$\tau^* + \frac{\gamma^*}{2} = \frac{\pi^{-\frac{1}{2}}}{r_1}\int_{r_1}^{r_2} r\,dr \int_{-\varphi_0}^{+\varphi_0} [\varepsilon_1(r)\,T_1 + \varepsilon_2(r)\,(T_3 - T_1)]\,d\varphi \tag{2.30}$$

$$\gamma^* - \frac{2\alpha - 3}{1 - \alpha}\tau^* - \frac{\alpha}{1 - \alpha}\tau = \frac{\pi^{-\frac{1}{2}}}{r_1}\int_{r_1}^{r_2} r\,dr \int_{-\varphi_0}^{+\varphi_0} \times [\varepsilon_1(r)\,(T_3 + \tfrac{1}{2}T_1) + \varepsilon_2(r)\,(T_5 - \tfrac{1}{2}T_3)]\,d\varphi. \tag{2.31}$$

Here the argument of the T_n functions is given by equation (2.28), and we have written $\tau = (T_1 - T_2)/T_2$.

Now we can easily evaluate the heat flux vector in terms of α, τ, τ^* and find

$$\frac{q}{q_{fm}} = \frac{1 - \tau^*/\tau}{1 - \alpha} \tag{2.32}$$

where q_{fm} is the free-molecular value given by

$$q_{fm} = \rho_2 \alpha\tau/(2\sqrt{\pi}). \tag{2.33}$$

Here ρ_2 denotes the density appearing in the Maxwellian of the molecules remitted by the outer cylinder.

Let us consider now the functional (2.20) where of course $\tilde{\varepsilon}$, $\mathfrak{S}$ and $\mathbf{S}$ are those defined in this section, and

$$\left.\begin{aligned} \alpha_{33} &= r_1\pi^{\frac{1}{2}} \\ \alpha_{34} &= 2r_1\pi^{\frac{1}{2}} \\ \alpha_{44} &= -2r_1\pi^{\frac{1}{2}}\frac{2\alpha - 3}{1 - \alpha} \\ c_4 &= 2r_1\pi^{\frac{1}{2}}\tau\frac{\alpha}{1 - \alpha} \end{aligned}\right\} \tag{2.34}$$

while the scalar product is now

$$(f, g) = 2\pi\int_{r_1}^{r_2} f(r)\,g(r)\,r\,dr. \tag{2.35}$$

It is easily verified that J attains its minimum value when

$$\tilde{\varepsilon} = \varepsilon, \tilde{\gamma}^* = \gamma, \tilde{\tau}^* = \tau,$$

where ε, γ^*, τ^* constitute the solution of the system formed by equations (2.10, 2.30, 2.31).

The corresponding minimum value of J is easily shown to be related to the ratio q/q_{fm} given by equation (2.32) as follows:

$$q/q_{fm} = \frac{1}{1 - \alpha}\left[1 + \frac{\min J}{\tau c_4}\right]. \tag{2.36}$$

Then equation (2.36) is valid for any $\alpha \neq 1$. The case of the complete accommodation requires a special treatment. In this case we have $\tau^* = \tau$ and we can follow the same procedure already followed for the case $\alpha = 1$; so the constant γ^*

is determined by equation (2.30), while equation (2.31) is now meaningless.

Then, let us consider the following functional

$$J(\tilde{\varepsilon}, \tilde{\gamma}^*) = (\tilde{\varepsilon}, \tilde{\varepsilon} - \mathfrak{S}\,\tilde{\varepsilon} - 2\mathbf{S}) + \sqrt{(\pi)}\, r_1 \tilde{\gamma}^{*2} - 4\tau \sqrt{(\pi) r_1 \tilde{\gamma}^*} \qquad (2.37)$$

whose minimum, attained for $\tilde{\varepsilon} = \varepsilon$, $\tilde{\gamma}^* = \gamma^*$ is related to the heat flux as follows:

$$\frac{q}{q_{fm}} = 3 + \frac{\min J}{2\tau^2 \sqrt{\pi}\, r_1}. \qquad (2.38)$$

3. EXPLICIT CALCULATIONS AND COMPARISONS

(a) *Plane case*

Let us choose the following simple trial functions for $\tilde{\varepsilon}, \tilde{\gamma}^*, \tilde{\tau}^*$:

$$\tilde{\varepsilon}(x) = \begin{vmatrix} a_1 x \\ (\tfrac{3}{2})^{\frac{1}{2}}\, a_2 x \end{vmatrix} \qquad \tilde{\gamma}^* = a_3 \qquad \tilde{\tau}^* = a_4. \qquad (3.1)$$

If we denote by $\mathbf{a}$ the vector with components a_i $(i = 1, 2, 3, 4)$, $\mathbf{A}$ the matrix with elements $\alpha_{ik} = \alpha_{ki}$ $(i, k = 1, 2, 3, 4)$ and $\mathbf{c}$ the vector with components c_i $(i = 1, 2, 3, 4)$, we can write

$$J(\mathbf{a}) = \mathbf{a} \,.\, \mathbf{A}\mathbf{a} - 2\mathbf{a} \,.\, \mathbf{c} \qquad (3.2)$$

where we have put

$$c_1 = c_2 = c_3 = 0$$

$$\alpha_{11} = \pi^{-\frac{1}{2}} \left[\left(\frac{\delta^2}{4} - 1 \right) + \delta T_0(\delta) + \left(\frac{\delta^2}{2} + 2 \right) T_1(\delta) + 2\delta T_2(\delta) \right]$$

$$\alpha_{22} = \pi^{-\frac{1}{2}} \left[\left(\frac{9}{16}\delta^2 - \frac{21}{4} \right) + \frac{\delta}{4}(3\delta^2 + 21)\, T_0(\delta) + \frac{1}{2}\left(\frac{33}{4}\delta^2 + 21 \right) T_1(\delta) + \frac{\delta}{2}\left(\frac{\delta^2}{2} + 21 \right) T_2(\delta) \right]$$

$$\alpha_{12} = \pi^{-\frac{1}{2}} \left[\left(\frac{\delta^2}{8} - \frac{3}{2} \right) + \frac{\delta}{2}\left(\frac{\delta^2}{2} + 3 \right) T_0(\delta) + \left(\frac{5}{4}\delta^2 + 3 \right) T_1(\delta) + 3\delta T_2(\delta) \right]$$

$$\alpha_{13} = \pi^{-\frac{1}{2}} \left[\left(\frac{\delta}{2} - \frac{\pi^{\frac{1}{2}}}{2} \right) + \delta T_1(\delta) + 2T_2(\delta) \right] \qquad (3.3)$$

$$\alpha_{23} = \pi^{-\frac{1}{2}} \left[\left(\frac{\delta}{4} - \frac{\pi^{\frac{1}{2}}}{2} \right) + \frac{\delta^2}{2} T_0(\delta) + \tfrac{3}{2}\delta(T_1(\delta) + 2T_2(\delta) \right]$$

$$\alpha_{14} = \pi^{-\frac{1}{2}} \left[\left(\delta - \frac{5}{4}\pi^{\frac{1}{2}} \right) + \frac{\delta^2}{2} T_0(\delta) + 3\delta T_1(\delta) + 5T_2(\delta) \right]$$

$$\alpha_{24} = \pi^{-\frac{1}{2}} \left[\left(\frac{3}{2}\delta - \frac{5}{2}\pi^{\frac{1}{2}} \right) + \frac{7}{4}\delta^2 T_0(\delta) + 7\delta T_1(\delta) + \left(\frac{\delta^2}{2} + 10 \right) T_2(\delta) \right]$$

and c_4, α_{33}, α_{44}, α_{34} have been already defined in equation (2.21). If we minimize $J(\mathbf{a})$ with respect to $\mathbf{a}$ we obtain

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{c} \qquad (3.4)$$

$$\min J = -\mathbf{a} \,.\, \mathbf{c} = -\mathbf{c} \,.\, \mathbf{A}^{-1} \,.\, \mathbf{c}. \qquad (3.5)$$

The numerical results are given in Table 1 for the values $\alpha = 1$, $\alpha = 0{\cdot}826$ and $\alpha = 0{\cdot}759$. These values were obtained for argon and nitrogen, respectively, on aluminum by Teagan [2]. Figure 1 shows a comparison between his experimental data and our results; the maximum deviation of the data from the theoretical curve is about 10 per cent. It is to be noted that almost all the experimental data lie above the theoretical curve: therefore the disagreement cannot depend upon the use of the variational method which always approximate from above (presumably with extreme accuracy) the exact solution according to BGK model. We also point out that the data are in better agreement with the results by Liu and Lees based on a moment method than with our solution (theoretically more accurate). This means that either the BGK

model corrects the Liu and Lees simple result in a wrong way or the experimental data contain a systematic error which results in larger values for q/q_{fm}.

(b) *Cylindrical case*

Let us choose the following simple trial functions for $\tilde{\varepsilon}$, $\tilde{\gamma}^*$, $\tilde{\tau}^*$:

$$\tilde{\varepsilon}(r) = \begin{vmatrix} a_1 \log(r/r_2) \\ (\tfrac{3}{2})^{\frac{1}{2}}\, a_2 \log(r/r_2) \end{vmatrix} \qquad \tilde{\gamma}^* = a_3, \qquad \tau^* = a_4. \tag{3.6}$$

We obtain again equation (3.2) for J (**a**) where now we have put

$$c_1 = c_2 = c_3 = 0$$

$$\alpha_{11} = 2\pi \int_{r_1}^{r_2} r \log^2(r/r_2)\, \mathrm{d}r - 2 \int_{r_1}^{r_2} r \log(r/r_2)\, \mathrm{d}r \int_{\Sigma(\mathbf{x})} \log(r'/r_2)\, T_0 \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$$\alpha_{22} = 3\pi \int_{r_1}^{r_2} r \log^2(r/r_2)\, \mathrm{d}r - 2 \int_{r_1}^{r_2} r \log(r/r_2)\, \mathrm{d}r \int_{\Sigma(\mathbf{x})} \log(r'/r_2)\, (T_4 - 2T_2 + \tfrac{3}{2}T_0) \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$$\alpha_{12} = -2 \int_{r_1}^{r_2} r \log(r/r_2)\, \mathrm{d}r \int_{\Sigma(\mathbf{x})} \log(r'/r_2)\, (T_2 - T_0) \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

Here the arguments of the T_n functions are the same as in equation (2.27).

$$\alpha_{13} = -2 \int_{r_1}^{r_2} \log(r/r_2)\, r\, \mathrm{d}r \int_{-\varphi_0}^{+\varphi_0} T_1\, \mathrm{d}\varphi$$

$$\alpha_{23} = -2 \int_{r_1}^{r_2} \log(r/r_2)\, r\, \mathrm{d}r \int_{-\varphi_0}^{+\varphi_0} (T_3 - T_1)\, \mathrm{d}\varphi$$

$$\alpha_{14} = 2 \int_{r_1}^{r_2} \log(r/r_2)\, r\, \mathrm{d}r \int_{-\varphi_0}^{+\varphi_0} (T_3 + \tfrac{1}{2}T_1)\, \mathrm{d}\varphi$$

$$\alpha_{24} = 2 \int_{r_1}^{r_2} \log(r/r_2)\, r\, \mathrm{d}r \int_{-\varphi_0}^{+\varphi_0} (T_5 - \tfrac{1}{2}T_3)\, \mathrm{d}\varphi.$$

Here the arguments of the T_n functions are the same as in equation (2.26). The other coefficients, i.e. c_4, α_{33}, α_{34}, α_{44} have been already defined in equations (2.34). Equations (3.4) and (3.5) again hold and the results for q/q_{fm} are given in Table 2 for $\alpha = 0{\cdot}7328$, $0{\cdot}4670$, $0{\cdot}2831$ and $r_1/r_2 = 0{\cdot}0030598$. These values correspond to the data of Dybbs and Springer [10] for argon, neon and helium, respectively, on tungsten. The values for q/q_{fm} according to the solution given by Hurlbut [9] on the basis of Liu and Lees' method is also reported; as is seen, the latter values differ from our results at most by 5 per cent. Since the deviations of the data by Dybbs and Springer from Hurlbut's solution are within the claimed experimental errors, we have an idea of the agreement with

Table 1. Variational results for the heat flux in the plane case

δ	$\alpha = 1$	$\alpha = 0{\cdot}826$	$\alpha = 0{\cdot}759$
0·01	0·9925	0·9947	0·9954
0·1	0·9352	0·9535	0·9593
0·5	0·7683	0·8250	0·8443
1·0	0·6409	0·7173	0·7448
1·25	0·5939	0·6751	0·7051
1·5	0·5539	0·6383	0·6700
1·75	0·5194	0·6058	0·6387
2·0	0·4894	0·5767	0·6104
2·5	0·4391	0·5266	0·5614
3·0	0·3986	0·4850	0·5200
4·0	0·3370	0·4194	0·4539
5·0	0·2922	0·3698	0·4030
7·0	0·2311	0·2993	0·3295
10·0	0·1760	0·2329	0·2588

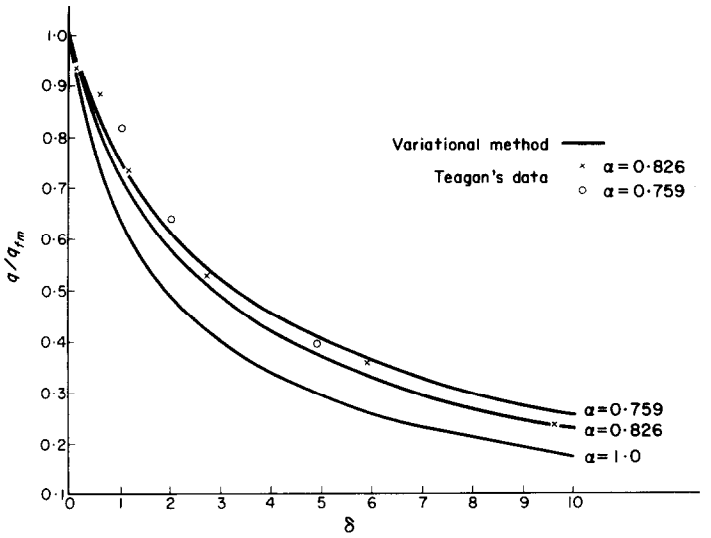

FIG. 1. Comparison between Teagan's data and our results.

the experimental data from Table 2. In this case we note that one can have a further improvement of the agreement, because the exact solution must be smaller than the variational results; the corrections, however, are expected to be of order 0·5 per cent.

In the case $\alpha = 1$, we use the same trial functions for ε, γ^*; $\tau^* = \tau$ is no longer an unknown. Accordingly we obtain equation (3.2) again where now **a** and **c** are three-dimensional vectors and **A** a three-by-three matrix. The new matrix has the elements α_{ik} $(i, k = 1, 2, 3)$ of the previous one and $c_i = -\alpha_{i4} \cdot \tau$ $(i = 1, 2, 3)$. Equations (3.4) and (3.5) again hold and the numerical results are given in Table 3 for different values of r_1/r_2 (0·4760077, 0·2024490, 0·007074830, 0·0030598, $0{\cdot}78 \times 10^{-4}$).

The choice of these values has been suggested

Table 2. Comparison between the moment method and the variational results for the heat flux in the cylindrical case ($r_2/r_2 = 0{\cdot}0030598$)

r_1	$\alpha = 0{\cdot}7328$		$\alpha = 0{\cdot}4670$		$\alpha = 0{\cdot}2831$	
	Variational method	Movement method	Variational method	Moment method	Variational method	Moment method
0·05	0·9391	0·9126	0·9603	0·9425	0·9756	0·9643
0·075	0·9065	0·8744	0·9384	0·9162	0·9617	0·9474
0·1	0·8749	0·8393	0·9165	0·8912	0·9477	0·9311
0·2	0·7634	0·7231	0·8350	0·8038	0·8930	0·8711
0·3	0·6743	0·6351	0·7646	0·7320	0·8427	0·8184
0·4	0·6029	0·5663	0·7043	0·6720	0·7971	0·7717
0·5	0·5431	0·5109	0·6509	0·6211	0·7547	0·7300
0·75	0·4357	0·4105	0·5478	0·5221	0·6664	0·6432
1·0	0·3632	0·3431	0·4723	0·4504	0·5962	0·5748
1·5	0·2721	0·2582	0·3696	0·3533	0·4918	0·4740
2·0	0·2173	0·2071	0·3034	0·2906	0·4181	0·4033
3.0	0·1548	0·1483	0·2232	0·2146	0·3214	0·3106
5·0	0·0980	0·0946	0·1456	0·1408	0·2193	0·2128
7·0	0·0716	0·0694	0·1079	0·1048	0·1662	0·1619
10·0	0·0507	0·0496	0·0774	0·0757	0·1215	0·1191

Table 3. Variational results for the heat flux in the cylindrical case ($\alpha = 1$)

r_1	$r_1/r_2 =$ 0·4760077	$r_1/r_2 =$ 0·2024490	$r_1/r_2 =$ 0·00707483	$r_1/r_2 =$ 0·0030598	$r_1/r_2 =$ $0{\cdot}78\cdot10^{-4}$
0·05	0·9866	0·9733	0·9175	0·8874	0·8589
0·075	0·9800	0·9602	0·8955	0·8490	
0·1	0·9734	0·9473	0·8481	0·8143	0·7428
0·2	0·9477	0·8984	0·7348	0·6946	
0·3	0·9230	0·8535	0·6431	0·5994	0·4614
0·4	0·8993	0·8123	0·5692	0·5345	
0·5	0·8766	0·7745	0·5094	0·4650	0·3361
0·75	0·8242	0·6924	0·4016	0·3608	
1·0	0·7771	0·6246	0·3307	0·2941	0·1981
1·5	0·6966	0·5200	0·2436	0·2141	
2·0	0·6303	0·4436	0·1924	0·1680	
3·0	0·5281	0·3408	0·1351	0·1172	0·0745
5·0	0·3962	0·2309	0·0845	0·0729	0·0455
7·0	0·3157	0·1740	0·0617	0·0533	
10·0	0·2411	0·1268	0·0436	0·0374	0·0231
20·0	0·1340	0·0665	0·0221	0·0190	

by the experimental works by Takao [11], Schäfer *et al.* [12], Dybbs and Springer [10], and Bömelburg [13], although a direct comparison between the data of these researchers and ours is not possible because either α is different from one or is not experimentally known.

Finally Table 4 and Fig. 2 show a comparison

Table 4. Comparison between variational results and Anderson's nonlinear calculations for the heat flux in the cylindrical case ($\alpha = 1$, $r_1/r_2 = 0{\cdot}5$)

r_1	Variational results	Anderson's results $\tau = 0{\cdot}1$	Anderson's results $\tau = 1$
0·007071	0·9983	0·998	0·998
0·07071	0·9823	0·988	0·986
0·7071	0·8415	0·826	0·822
1·4142	0·7225	0·708	0·698
2·8284	0·5598	0·542	0·531
5·6568	0·3818	0·370	0·361

between our results for $\alpha = 1$, $r_1/r_2 = 0{\cdot}5$ and the results obtained by Anderson [14] through a numerical solution of the nonlinear BGK model. Accordingly, Anderson's results depend upon τ as shown in Table 4 ($\tau = 0{\cdot}1$, 1); however, the dependence is slight and the results are well approximated (maximum error 3·5 per cent for $\tau = 0{\cdot}1$, 5 per cent for $\tau = 1$) by the variational calculations on the linearized model.

4. CONCLUDING REMARKS

The variational method used in this paper for the BGK model and arbitrary accommodation coefficients is believed to give essentially correct results for the heat transfer between parallel plates and concentric cylinders. The agreement with available experimental results is good but not as good as for the moment method. This fact can be due either to an inaccuracy of the experimental data or of the BGK model. In order to find the proper answer to this question, solutions of more sophisticated models as well as more accurate measurements are required. In particular the experimental values for α do not seem very accurate and this could influence the comparison between theory and experiment in a considerable way.

After the paper was submitted to this Journal, a referee suggested that we compared our treatment with a recent work by Su [15]. Su presented a variational principle for the heat flux based on variations of density and temperature only. Both Su and the referee lay stress on this fact as if it were a simplification; actually,

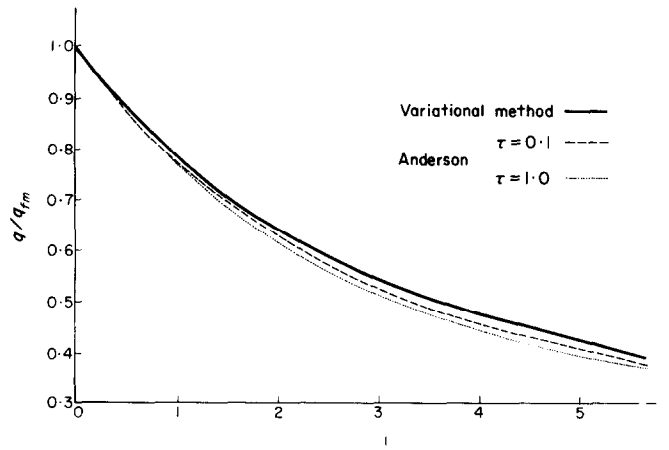

FIG. 2. Comparison between our solution and Anderson's nonlinear results.

Su's variational principle can be obtained from ours by elimination of γ^* and τ^* through equations (2.30) and (2.31). Also, our method is more flexible since it allows variations of γ^* and τ^* which violate equations (2.30) and (2.31), with the consequence of obtaining heat fluxes closer to the correct ones, with about the same amount of calculations. Su did not use his variational principle for actual computations, but paid considerable attention to the limiting solutions when the Knudsen number, based on the inner cylinder radius, was large. A comparison of his results with ours and with experiments would be quite interesting. However, it would lead us outside the aim and the size of the present paper.

REFERENCES

1. P. BASSANINI, C. CERCIGNANI and C. D. PAGANI, Comparison of kinetic theory analyses of linearized heat transfer between parallel plates, *Int. J. Heat Mass Transfer* **10**, 447 (1967).
2. W. P. TEAGAN, Ph.D. Thesis, Massachusetts Institute of Technology (1967).
3. C. CERCIGNANI and C. D. PAGANI, Variational approach to boundary-value problems in kinetic theory, *Physics Fluids* **9**, 1167 (1966).
4. C. CERCIGNANI and C. D. PAGANI, Variational approach to rarefied flows in cylindrical and spherical geometry, in *Rarefied Gas Dynamics*, edited by C. L. BRUNDIN, Vol. 1, p. 555. Academic Press, New York (1967).
5. C. CERCIGNANI, Boundary value problems in linearized kinetic theory, in Proceedings of the Symposium on Transport Theory, New York (1967).
6. P. WELANDER, On the temperature jump in a rarefied gas, *Ark. Fys.* **7**, 507 (1954).
7. S. NOCILLA, The Surface re-emission law in free-molecule flow, in *Rarefied Gas Dynamics*, edited by J. A. LAURMANN, Vol. 1, p. 327. Academic Press, New York (1963).
8. F. C. HURLBUT, Current developments in the study of gas-surface interactions, in *Rarefied Gas Dynamics*, edited by C. L. BRUNDIN, Vol. 1, p. 1. Academic Press, New York (1967).
9. F. C. HURLBUT, Note on conductive heat transfer from a fine wire, *Physics Fluids* **7**, 904 (1964).
10. A. DYBBS and G. S. SPRINGER, Heat conduction experiments in rarefied gases between concentric cylinders, *Physics Fluids* **8**, 1946 (1965).
11. K. TAKAO, Heat conduction in rarefied gases, *Mem. Defense Acad. Japan* **2**, 9 (1962).
12. K. SCHÄFER, W. RATING and A. EUCKEN, Über den Einfluss des gehemmten Austauschs der Translations— und Schwingungsenergie auf das Wärmeleitvermögen der Gase, *Ann. Phys.* **42**, 176 (1964).
13. H. J. BÖMELBURG, Heat loss from very thin heated wires in rarefied gases, *Physics Fluids* **2**, 717 (1959).
14. D. G. ANDERSON, On the steady Krook kinetic equation: II, Harvard University, Engineering Sciences Laboratory, Technical Report No. 27 (1966).
15. C. L. SU, Heat conduction in a rarefied gas between concentric cylinders, Berkeley University, College of Engineering, Report No. AS-67-15 (1967).

Résumé—On considère le transport de chaleur linéarisé entre des plaques parallèles et des cylindres concentriques pour des inverses du nombre de Knudsen compris entre 0 et 10, et des coefficients d'accom-

modation arbitraires. Le modèle de Bhatnagar, Gross et Krook est employé et transformé en un système d'équations intégrales.

Ces équations sont résolues par une méthode variationnelle appliquée auparavant à des situations plus simples. On compare ces solutions avec les résultats expérimentaux et des calculs obtenus auparavant.

Zusammenfassung—Es wird der linearisierte Wärmeübergang zwischen parallelen Platten und konzentrischen Zylindern untersucht für Kehwerte der Knudsen-Zahl von 0–10 und beliebigen Akkomodationskoeffizienten. Das Modell nach Bhatnagar, Gross und Krook wird herangezogen und in ein System von Integralgleichungen transformiert. Diese Gleichungen werden nach Variationsmethoden gelöst, die kürzlich für einfachere Gegebenheiten angewandt wurden. Vergleiche wurden gezogen zu Versuchswerten und kürzlich erhaltenen Berechnungen.

Аннотация—Рассматривается линеаризованный теплообмен между параллельными пластинами и концентрическими цилиндрами для обратных значений чисел Кнудсена в диапазоне изменения от 0 до 10 и для произвольных коэффициентов аккомодации. Используется модель Батнадара, Гросса и Крука и затем преобразуется в систему интегральных уравнений.

Эти уравнения решаются вариационным методом, который ранее применялся к более простым случаям. Экспериментальные результаты сравниваются с имеющимися в литературе расчетами.